tion as to whether one is justified in using a more detailed correlation than Eq. (3):

$$q/q_0 = 1.4Ze^{-Z} \qquad (Z > 1) \quad (3)$$

In any case, the principal long-term value of these and related data may well be other than the establishment of mass-transfer cooling levels. In the long view, they may be even more valuable as a new type of observation of the attributes of turbulence near a surface.

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# Method of Characteristics and Velocity of Sound for Reacting Gases

F. Edward Ehlers*
*Boeing Scientific Research Laboratories, Seattle, Wash.*

Although there is an evident disagreement between the definition of the sound speed in a reacting gas given by Resler,[5–7] and that found by Kirkwood, Chu, and others, the reason for this discrepancy has not been made clear. In this note, the author will show how Resler's analysis may be revised to become consistent with the classical analysis of the method of characteristics. Accordingly Eqs. (19) and (20) of Resler's paper,[7] describing the two-dimensional steady flow of a gas, will be considered. These may be written

$$\theta_n + \rho_s/\rho + w_s/w = 0 \quad (1)$$

$$\theta_s + p_n/\rho w^2 = 0 \quad (2)$$

$$ww_s + p_s/\rho = 0 \quad (3)$$

where $\theta$ is the flow angle, $w$ the velocity, $p$ the pressure, and the subscripts $s$ and $n$ denote differentiation with respect to the stream direction and its normal, respectively. Eliminating $w_s$ between Eqs. (1) and (3) and introducing the directional derivative

$$\theta' = \theta_s + \theta_n/\beta \quad (4)$$

where $\beta = (M^2 - 1)^{1/2}$, $M = w/a$, and $a$ is to be determined, Eqs. (1–3) are combined to give

$$\theta' + \beta p'/\rho w^2 = 1/\beta\rho(p_s/a^2 - \rho_s) \quad (5)$$

Since Eq. (5) must contain only primed derivatives when $dn/ds = 1/\beta$ is a characteristic direction, Resler set the right-hand side equal to zero in order to define the velocity of sound and obtained $a = (p_s/\rho_s)^{1/2}$.

Now consider the classical approach in which the characteristics are regarded as surfaces along which the derivatives are indeterminate. Assume further that the pressure is given by the simple functional relation

$$p = p(\rho, \eta) \quad (6)$$

Received January 4, 1963.
* Staff Member, Mathematics Research Laboratory.

with $\eta$ satisfying a reaction rate equation of the form

$$w\eta_s = \varphi \quad (7)$$

where $\varphi$ does not depend on derivatives of the flow quantities. After eliminating $w_s$ from Eqs. (1) by Eq. (3) and using Eqs. (6) and (7), one obtains

$$\theta_n + \beta^2\sigma_s/M^2 = a_1\varphi/w \quad (8)$$

$$\theta_s + \sigma_n/M^2 + a_1\eta_n = 0 \quad (9)$$

where $M^2 = w^2/(\partial p/\partial\rho)$ and $a_1 = (\partial p/\partial\eta)/\rho w^2$ have been substituted, and $\sigma = \log\rho$.

Let the equation for the characteristic be given by $n = n(s)$ Then Eqs. (8) and (9), with the conditions

$$\sigma_s + \sigma_n n' = \sigma' \quad (10)$$

$$\theta_s + \theta_n n' = \theta' \quad (11)$$

$$\eta_n n' = \eta' - \varphi/w \quad (12)$$

constitute a system of linear equations in the derivatives of $\sigma, \theta$, and $\eta$. By using Kramer's rule and applying the condition of indeterminancy, the differential equations for the characteristics are obtained as

$$n' = 0 \qquad n' = \pm 1/\beta$$

along which are the compatibility conditions,

$$\eta' = \varphi/w \quad (13)$$

and

$$\pm\theta' + \beta(a_1\eta' + \sigma'/M^2) = a_1M^2\varphi/\beta w \quad (14)$$

respectively. Combination of the partial derivatives of $p$ in Eq. (14) gives

$$\pm\theta' + \beta p'/\rho w^2 = (\varphi/\rho a^2\beta w)(\partial p/\partial\eta) \quad (15)$$

The significant difference between Eq. (15) and Resler's corresponding result is the inhomogeneous term on the right-hand side. Now reconsider Eq. (5) and assume for $p$, the pressure relation of Eq. (6). After using Eq. (7) and simplifying, the right-hand side of Eq. (5) becomes

$$(1/\rho\beta)[(\partial p/\partial\rho)/a^2 - 1](\partial p/\partial s) + (\varphi/\rho a^2\beta w)(\partial p/\partial\eta)$$

In order for the prime differentiation to be in the characteristic direction, the coefficient of $\rho_s$ must vanish. Thus is obtained

$$a^2 = (\partial p/\partial\rho)\eta$$

which is the "frozen" velocity of sound. In addition the inhomogeneous term in agreement with Eq. (15) is obtained. Thus it appears that setting the right-hand side of Eq. (5) equal to zero to define the velocity of sound is too severe a restriction.

From the preceding analysis the following conclusions can be drawn:

1) The introduction of finite reaction rate equations leads to the addition of inhomogeneous terms to the differential equations along the characteristic directions.

2) Contrary to Resler's assumption, it is not possible to define the velocity of sound using the equations of continuity and momentum alone. The reaction rate equations as well as the pressure relation must be used.

3) The velocity of sound for the method of characteristics is always that calculated for frozen flow conditions in agreement with Refs. 1–3 and 8–10.

One reason that the frozen speed of sound is not observed frequently is that the medium with finite reaction rates is dispersive and the bulk of the energy may not coincide with the wave front. The highest frequencies propagate with a velocity near the frozen speed of sound and the lowest frequencies with a velocity near the equilibrium speed of sound.

Since the frozen speed of sound usually is greater than the equilibrium value, the frozen speed becomes identified with the wave front (see Moore and Gibson[12, 13]).


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# Surface Temperatures Due to Localized Removal of a High-Emittance Coating on the Thin-Plate Sections of a Re-Entry Vehicle


Fritz R. S. Dressler*

*Ballistic Research Laboratories, Aberdeen Proving Ground, Md.*


### Nomenclature

$T$ = temperature
$T_{ab}$ = adiabatic wall temperature
$T_j$ = equilibrium temperature appropriate to $\epsilon_j (j = 1,2)$
$r$ = length of radius vector
$\epsilon$ = emittance
$H$ = heat transfer coefficient
$k$ = conductivity of plate
$t$ = plate thickness (necessarily small)
$\sigma$ = Stefan-Boltzman constant
$\beta_j$ = $[4\epsilon_j\sigma T_j^3 + H_j]/kt\ (j = 1,2)$
$C_1$ = $K_1/[K_1I_0 + (\beta_1/\beta_2)^{1/2}K_0I_1]$
$C_2$ = $I_1/[K_0I_1 + (\beta_2/\beta_1)^{1/2}K_1I_0]$
$C_3$ = $1/\{\cosh[(\beta_1)^{1/2}S] + (\beta_1/\beta_2)^{1/2}\sinh[(\beta_1)^{1/2}S]\}$
$C_4$ = $1/\{1 + (\beta_2/\beta_1)^{1/2}\coth[(\beta_1)^{1/2}S]\}$
$I_0$ = modified Bessel function of first kind, zero order
$I_1$ = modified Bessel function of first kind, first order
$K_0$ = modified Bessel function of second kind, zero order
$K_1$ = modified Bessel function of second kind, first order
$I$'s have the argument $[(\beta_1)^{1/2}R]$; the $K$'s, $[(\beta_2)^{1/2}R]$


Received February 11, 1963.


* Mathematician.

CONSIDER an element of surface $\Delta A$ exposed to high-speed flow. The fundamental equations are developed by noting the following: 1) energy radiated away from the element $= \epsilon\sigma T^4\Delta A$; 2) energy convected to the surface $= H(T_{ab} - T)\Delta A$; 3) energy conducted into the volume beneath $\Delta A$ (the temperature through the plate is essentially constant) $= kt\nabla^2T\Delta A$.

Consider steady-state conditions: i.e., $\mathring{T} \equiv 0$. The foregoing considerations give

$$\nabla^2T = (\epsilon\sigma/kt)T^4 - (H/kt)(T_{ab} - T) \quad (1)$$

Let a small circular area of radius $r = R$ have a lower emittance than the rest of the plate. Now suppose for a moment that no energy is transmitted (conducted) between this low emittance area $0 \le r \le R$ and the rest of the plate. The temperature on the plate is $T_2$ and, on the low emittance area, $T_1(\epsilon_1 < \epsilon_2)$. When $(T_1 - T_2)/T_j\ (j = 1,2)$ is small, it is said that, if energy were allowed to be transmitted, the new steady-state temperature profile that would establish itself could not differ greatly from the previous equilibrium temperature profile. In view of this, one can write

$$T = T_j + T_j' \quad (j = 1,2)$$

where $T_j'$ is a small quantity with respect to $T_j$. Incorporating the foregoing discussion into Eq. (1),

$$\nabla^2T_j' = (1/kt)(4\sigma\epsilon_jT_j^3 + H_j)T_j' \quad (2)$$

where terms of order $(T_j'/T_j)^2$ and higher have been neglected.

Based on a polar coordinate system placed at the center of the circular area, Eq. (2) is solved separately in region $0 \le r \le R(j = 1)$ and in region $r > R(j = 2)$. The separate solutions then are matched at $r = R$. The resulting solutions, satisfying all boundary conditions, can by given. For $0 \le r \le R$,

$$T = T_1 - C_1(T_1 - T_2)I_0[(\beta_1)^{1/2}r] \quad (3)$$

For $r > R$,

$$T = T_2 + C_2(T_1 - T_2)K_0[(\beta_2)^{1/2}r] \quad (4)$$

The equation of interest is Eq. (3). It is found that, in all applications, the argument of $I_0$ is small, i.e., $I_0 \approx 1$ for values of $0 \le r \le R$. This permits one to write

$$T \approx T(0) = T_1 - C_1(T_1 - T_2) \quad (5)$$

for the hot circular area.

Equation (2) may be used also in the case of a long, narrow (length/width $\gg 1$) area of low emittance. Based on a rectangular Cartesian coordinate system located at the center of the long strip ($y$ axis along the strip), Eq. (2) is solved similarly to the polar case. The resulting solutions, satisfying all boundary conditions, can be given. For $0 \le x \le S$ = semiwidth of strip,

$$T = T_1 - C_3(T_1 - T_2)\cosh[(\beta_1)^{1/2}x] \quad (6)$$

For $x > S$,

$$T = T_2 + C_4(T_1 - T_2)\exp[(\beta_2)^{1/2}(S - x)] \quad (7)$$

The equation of interest is Eq. (6). The temperature of the strip can be approximated by

$$T \approx T(0) = T_1 - C_3(T_1 - T_2) \quad (8)$$

A comparison of Eqs. (5) and (8) reveals that, for similar conditions, the narrow strip is critical. In other words, a long scratch of width $2S$ will be hotter than a circular spot of diameter $2R$ where $S = R$.